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REGENERATIVE SIMULATION OF RESPONSE TIMES IN NETWORKS OF QUEUES--ETC(U)

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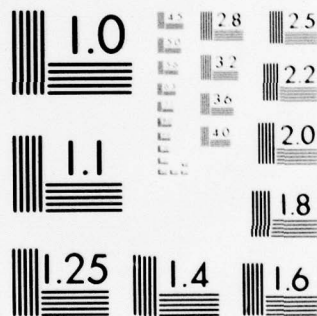
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REGENERATIVE SIMULATION OF RESPONSE TIMES  
IN NETWORKS OF QUEUES: STATISTICAL EFFICIENCY\*

by

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and

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## 1. INTRODUCTION

In previous papers (Iglehart and Shedler (1978a), (1978b), (1979) and Shedler (1979)), we provided methods for obtaining from a single simulation run point estimates and confidence intervals for general characteristics of "passage times" in certain closed networks of queues. Informally, a passage time is the time for a job to traverse a portion of the network. Such quantities are important in computer and communication system models, and in this context, quantities other than mean values are of interest. The basis for these estimation methods is the regenerative method for simulation analysis (Crane and Iglehart (1975)). For an introduction to and a detailed review of the regenerative method, see Crane and Lemoine (1977) and Iglehart (1975a).

We consider here the calculation of theoretical values for variance constants entering into the central limit theorems used to obtain confidence intervals from passage time simulations. Using results of Hordijk, Iglehart, and Schassberger (1976) for the calculation of moments in discrete time and continuous time Markov chains, we calculate variance constants pertinent to mean passage times. We do this first for the broadly applicable "marked job method" for passage time simulation which is based on the tracking of a distinguished job, and then for the "decomposition method" in which observed passage times for all of the jobs enter into the construction of point and interval estimates. The decomposition method provides point and interval estimates for a restricted but practically important class of passage times, namely, those

corresponding to the passage through a subnetwork of the given network of queues.

The results of this paper provide a firm basis for comparing the statistical efficiency of the two methods when both apply. The calculations also permit an assessment of the efficacy of the marked job method for simulation of "response times" (complete circuits in the network); the marked job method is apparently the only available means of obtaining confidence intervals for response times from a single simulation run.

## 2. PRELIMINARIES

We consider closed networks of queues with a finite number of jobs (customers),  $N$ . In each network there are a finite number of service centers,  $s$ , and a finite number of job classes,  $c$ . At each epoch of time each job is in exactly one job class, but jobs may change class as they traverse the network. Upon completion of service at center  $i$ , a job of class  $j$  goes to center  $k$  and changes to class  $l$  with probability  $p_{ij,kl}$ . We assume that  $P = (p_{ij,kl} : 1 \leq i, k \leq s, 1 \leq j, l \leq c)$  is a given irreducible Markov matrix.

At each service center jobs queue and receive service according to a fixed priority scheme among classes, which scheme can vary from center to center. Each center operates as a single server, processing jobs of a fixed class according to a fixed service discipline. All service times in the network are mutually independent, and at each center have a distribution with a Cox-phase (exponential stage) representation

(cf. Cox (1955), Gelenbe and Muntz (1976)) with parameters which may depend on the service center, class of job being serviced, and the "state" of the entire system. (We exclude zero service times occurring with positive probability.) A job in service may or may not be preempted (according to a fixed procedure for each center) if another job of higher priority joins the queue at the center. We restrict the present discussion to networks in which all service times are exponentially distributed, and deal with distributions having a Cox-phase representation in the usual way by the method of stages.

As in Iglehart and Shedler (1978a), we view the  $N$  jobs as being completely ordered in a linear stack, and define the vector  $Z(t)$  according to

$$Z(t) = \left( c_{j_{k(1)}}^{(1)}(t), \dots, c_{j_1}^{(1)}(t), S_1(t); \dots; \right. \\ \left. c_{j_{k(s)}}^{(s)}(t), \dots, c_{j_1}^{(s)}(t), S_s(t) \right). \quad (2.1)$$

The linear stack corresponds to the order of components in the vector  $Z(t)$  after ignoring any zero components. Within a class at a center, jobs waiting appear in the linear stack in the order of their arrival at the center, the latest to arrive being closest to the top of the stack.

Letting  $N(t)$  denote the position from the top of the marked job in this linear stack, for  $t \geq 0$  the state vector of the network is

$$X(t) = (Z(t), N(t)). \quad (2.2)$$



As before, we specify the passage time for the marked job by four subsets  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$  of  $E$ , the state space of the process  $X = \{X(t) : t \geq 0\}$ . The sets  $A_1$  and  $A_2$  [resp.  $B_1$  and  $B_2$ ] determine when to start [resp. stop] the clock measuring a particular passage time for the marked job. Denoting the jump times of  $X$  by  $\{\tau_n : n \geq 0\}$ , for  $k, n \geq 1$  we require that the sets  $A_1$ ,  $A_2$ ,  $B_1$  and  $B_2$  satisfy:

if  $X(\tau_{n-1}) \in A_1$ ,  $X(\tau_n) \in A_2$ ,  $X(\tau_{n-1+k}) \in A_1$  and  $X(\tau_{n+k}) \in A_2$ ,  
then  $X(\tau_{n-1+m}) \in B_1$  and  $X(\tau_{n+m}) \in B_2$  for some  $0 < m \leq k$ ;

and

if  $X(\tau_{n-1}) \in B_1$ ,  $X(\tau_n) \in B_2$ ,  $X(\tau_{n-1+k}) \in B_1$  and  $X(\tau_{n+k}) \in B_2$ ,  
then  $X(\tau_{n-1+m}) \in A_1$  and  $X(\tau_{n+m}) \in A_2$  for some  $0 < m \leq k$ .

These conditions ensure that the start and termination times for the specified passage time strictly alternate. Also in terms of these jump times, we define two sequences of random times:  $\{S_j : j \geq 0\}$  and  $\{T_j : j \geq 1\}$ . The start [resp. termination] time of the  $j$ th passage time for the marked job is denoted by  $S_{j-1}$  [resp.  $T_j$ ]. Assuming that a passage time for the marked job begins at  $t=0$ , we have

$$S_0 = 0$$

$$S_j = \inf\{n \geq T_j : X(\tau_n) \in A_1, X(\tau_{n-1}) \in A_2\}, \quad j \geq 1$$

$$T_j = \inf\{n \geq S_{j-1} : X(\tau_n) \in B_1, X(\tau_{n-1}) \in B_2\}, \quad j \geq 1.$$

Then the  $j$ th passage time for the marked job is  $P_j = T_j - S_{j-1}$ ,  $j \geq 1$ . Note that these definitions are somewhat more restrictive than those in [10].

We let  $X_n$  denote the state of the process  $X$  when the  $(n+1)$ st passage time of the marked job begins:  $X_n = X(S_n)$ ,  $n \geq 0$ . The process  $\{(X_n, P_{n+1}) : n \geq 0\}$  is a regenerative process in discrete time, and the regenerative property guarantees (Miller (1972)) that as  $n \rightarrow \infty$ ,

$$(X_n, P_{n+1}) \Rightarrow (X, P) ,$$

where  $\Rightarrow$  denotes convergence in distribution. The random variable  $P$  is the limiting passage time for the marked job, and the argument in the Appendix of [9] shows that the sequence of passage times for any other job also converges in distribution to the same random variable. The goal of the simulation is estimation of

$$r(f) = E\{f(P)\} , \quad (2.3)$$

where  $f$  is a real-valued measurable function with domain  $R_+ = [0, \infty)$ . We assume that  $P\{P \in D(f)\} = 0$ , where  $D(f)$  is the set of discontinuities of the function  $f$ .

### 3. THEORETICAL VALUES FOR FINITE STATE MARKOV CHAINS

Let  $\{X_k : k \geq 0\}$  be an irreducible Markov chain with finite state space  $E = \{0, 1, \dots, N\}$  and one-step transition matrix

$$P = \{p_{ij} : i, j \in E\} .$$

For  $n \geq 1$ ,  $p_{ij}^n$  denotes the  $n$ -step transition probability from state  $i$  to state  $j$ , and

$$P^n = \{p_{ij}^n : i, j \in E\} .$$

For a fixed state  $i \in E$ ,  $P_i\{\cdot\}$  denotes the conditional probability associated with starting the chain in state  $i$ , and  $E_i\{\cdot\}$  denotes the corresponding conditional expectation. For  $j \in E$  and  $n \geq 1$ , we let  $\beta_n(j)$  denote the  $n$ th entrance time of  $\{X_k: k \geq 0\}$  to state  $j$ , e.g.,

$$\beta_1(j) = \min\{k \geq 1: X_k = j\}$$

and let  $\alpha_1(j) = \beta_1(j)$  and  $\alpha_n(j) = \beta_n(j) - \beta_{n-1}(j)$ ,  $n \geq 1$ .

We consider vectors such as  $(v(0), v(1), \dots, v(N))$  to be column vectors, and view real-valued functions, such as  $f$  and  $g$ , having domain  $E$  in this way. Unless specified otherwise, the symbol  $E\{\cdot\}$  denotes the vector

$$(E_0\{\cdot\}, E_1\{\cdot\}, \dots, E_N\{\cdot\}) .$$

In addition (for vectors  $u$  and  $v$ ) the symbol  $u \cdot v$  denotes the vector

$$(u(0)v(0), u(1)v(1), \dots, u(N)v(N)) .$$

For a matrix  $A = (a_0, a_1, \dots, a_m)$ , we let

$$u \cdot A = A \cdot u = (u \cdot a_0, u \cdot a_1, \dots, u \cdot a_m) .$$

Finally, for a matrix  $B = (b_0, b_1, \dots, b_m)$ , we let

$$A \cdot B = (a_0 \cdot b_0, a_1 \cdot b_1, \dots, a_m \cdot b_m) .$$

For the discrete time Markov chain  $\{X_k: k \geq 0\}$ , we consider here only cycles of the regenerative process formed by the successive entrances to state 0, and henceforth suppress the 0 in the notation  $\beta_n(0)$ ,  $\alpha_n(0)$ , etc. For  $i, j \in E$  and  $n = 0, 1, \dots$ , let



$${}_0P_{ij}^n = P_1(\alpha_1 > n, X_n = j) ,$$

and set

$${}_0P^n = \{{}_0P_{ij}^n : i, j \in E\} .$$

We obtain  ${}_0P^1 = {}_0P$  from  $P$  by setting the 0-column of  $P$  equal to 0. It is easy to see that  ${}_0P^n$  is the matrix product of  $n$  copies of  ${}_0P$ , and that for all  $n \geq 1$ ,  ${}_0P_{i0}^n = 0$ .

For any real-valued function  $f$  with domain  $E$ , the state space of  $\{X_k : k \geq 0\}$ , we define

$$Y_1(f) = \sum_{k=0}^{B_1-1} f(X_k) .$$

Theorem (3.1) of Hordijk, Iglehart, and Schassberger (1976) shows that for an irreducible, finite state discrete time Markov chain with transition matrix  $P$ ,

$$E\{Y_1(f)\} = (I - {}_0P)^{-1}f \quad (3.1)$$

and

$$E\{Y_1(f)Y_1(g)\} = (I - {}_0P)^{-1}h , \quad (3.2)$$

where  $h = f \circ E\{Y_1(g)\} + g \circ E\{Y_1(f)\} - f \cdot g$ .

Now we consider continuous time Markov chains  $X = \{X(t) : t \geq 0\}$  with finite state space  $E = \{0, 1, \dots, N\}$ , transition matrix  $P(t) = \{p_{ij}(t) : i, j \in E\}$  and matrix of infinitesimal parameters  $Q = \{q_{ij} : i, j \in E\}$ . The exponentially distributed

holding time in any state  $i \in E$  has rate parameter  $q_i = -q_{ii}$ . For all  $i \in E$ , we assume that  $0 < q_i < \infty$ , i.e., that all states are stable and nonabsorbing, and in addition that

$$\sum_{j=0}^N q_{ij} = 0.$$

This last assumption guarantees that, starting from any state  $i \in E$ , the Markov chain  $X$  makes a transition to a next state  $j \in E$ . We now form the jump matrix  $R = (r_{ij})$  of the chain, defining the elements  $r_{ij}$  according to

$$r_{ij} = \begin{cases} q_{ij}/q_i & \text{if } j \neq i \\ 0 & \text{if } j = i \end{cases}.$$

We assume that the jump matrix  $R$  is irreducible (and therefore positive recurrent). This is equivalent to the continuous time Markov chain  $X$  being irreducible. For  $j \in E$  and  $n \geq 1$ , we let  $\beta_n(j)$  denote the  $n$ th entrance time of  $X$  to state  $j$ , i.e.,

$$\beta_n(j) = \inf\{s > 0: X(s-) \neq j, X(s) = j\}.$$

As in the case of discrete time Markov chains, we restrict attention to regenerative cycles formed by the successive entrances to state 0, and suppress the 0 in our notation. For  $t \geq 0$ , we let

$${}_0P_{ij}(t) = P_1(\alpha_1 > t, X(t) = j),$$

$${}_0P(t) = ({}_0P_{ij}(t): i, j \in E),$$

and, for  $t \geq 0$ , construct the matrix  ${}_0P^h$  from  $R$  in the same manner as we constructed  ${}_0P^h$  from  $P$  in the discrete time case.

For a real-valued function  $f$  defined on  $E$ , we define  $Y_1(f)$  according to

$$Y_1(f) = \int_0^{\beta_1} f(X(t)) dt ,$$

and let  $q^{-1}$  be the column vector

$$q^{-1} = (q_0^{-1}, q_1^{-1}, \dots, q_N^{-1}) .$$

Theorem (3.10) of Hordijk, Iglehart, and Schassberger (1976) shows that for an irreducible, finite state continuous time Markov chain with jump matrix  $R$  and vector  $q$  of rate parameters for holding times,

$$E\{Y_1(f)\} = E\left\{\int_0^\infty P(t) f dt\right\} = (I - {}_0R)^{-1} (f \circ q^{-1}) \quad (3.3)$$

and

$$E\{Y_1(f)Y_1(g)\} = E\left\{\int_0^\infty \int_0^\infty P(t) h dt\right\} = (I - {}_0R)^{-1} (h \circ q^{-1}) \quad (3.4)$$

where  $h = f \circ E\{Y_1(g)\} + g \circ E\{Y_1(f)\}$ .

#### 4. VARIANCE CONSTANTS FOR THE MARKED JOB METHOD

We consider closed networks of queues and passage times as in Section 2. For  $t \geq 0$ , the state vector of the network is

$$X(t) = (Z(t), N(t)) ,$$

where  $Z(t)$  corresponds to the linear job stack of Equation (2.1), and  $N(t)$  is the position in the job stack of the marked job at time  $t$ . The process  $X = \{X(t) : t \geq 0\}$  is an irreducible, positive recurrent Markov chain

with state space  $E$ . Denoting by  $L(t)$  the last state visited by the Markov chain  $X$  before jumping to  $X(t)$ , the process  $\underline{V} = \{V(t) : t \geq 0\}$  where

$$V(t) = (L(t), X(t)) ,$$

is the fundamental stochastic process of the passage time simulation.

The process  $\underline{V}$  has a state space,  $F$ , consisting of all pairs of states  $(i, j)$ ,  $i, j \in E$  for which a transition in  $X$  from state  $i$  to state  $j$  can occur with positive probability. Since  $X$  is an irreducible, positive recurrent Markov chain, so is  $\underline{V}$ . We define subsets  $S$  and  $T$  of the state space  $F$  according to

$$S = \{(k, m) \in F : k \in A_1, m \in A_2\}$$

and

$$T = \{(k, m) \in F : k \in B_1, m \in B_2\} .$$

The entrances of  $\underline{V}$  to the fixed subset  $S$  [resp.  $T$ ] correspond to the starts [resp. terminations] of passage times for the marked job.

As in Iglehart and Shedlar (1973a), we select a (fixed) state of  $S$ , designated state 0, and assume that  $V(0) = 0$ . To estimate  $\tau(f)$ , the marked job method prescribes that we carry out the simulation of  $\underline{V}$  in 0-cycles defined by the successive returns to state 0; within each cycle we record the number of passage times of the marked job and measure each of these passage times.



We let  $\{V_n: n \geq 0\}$  denote the embedded jump chain associated with the continuous time process  $\underline{Y}$ . The random times  $\{\gamma_n: n \geq 1\}$  denote the lengths in discrete time units of the successive 0-cycles (successive returns to the fixed state 0) for  $\{V_n: n \geq 0\}$ , and we define  $\delta_0 = 0$  and  $\delta_m = \gamma_1 + \dots + \gamma_m$ ,  $m \geq 1$ . Then the number of passage times for the marked job in the first 0-cycle of  $\underline{Y}$  is

$$M_1 = \sum_{j=0}^{\delta_1-1} 1_{\{V_j \in S\}} ,$$

(For a set  $A$ ,  $1_A(x) = 1$  if  $x \in A$  and 0 if  $x \notin A$ . Here we suppress the argument  $\omega$ .) The sum of the values of the function  $f$  for the passage times of the marked job in that cycle is

$$Y_1(f) = \sum_{j=1}^{M_1} f(P_j) .$$

We denote the analogous quantities in the  $k$ th 0-cycle by  $M_k$  and  $Y_k(f)$ .

The key results leading to point estimates and confidence intervals for  $r(f)$  are that the pairs of random variables

$$\{(Y_k(f), M_k): k \geq 1\} \tag{4.1}$$

are independent and identically distributed, and that

$$r(f) = E_0\{Y_1(f)\}/E_0\{M_1\} , \tag{4.2}$$

provided that the quantity  $E\{|f(P)|\} < \infty$ .

Given Equations (4.1) and (4.2), the regenerative method provides from a fixed number  $n$  of 0-cycles the so-called classical point estimate (cf. Iglehart (1975b))

$$\hat{r}_n(f) = \bar{Y}_n(f) / \bar{M}_n,$$

and the associated confidence interval for  $r(f)$  follows from the central limit theorem

$$\frac{n^{1/2}(\hat{r}_n(f) - r(f))}{\sigma(f)/E_0(M_1)} \Rightarrow N(0,1). \quad (4.3)$$

Here  $N(0,1)$  is a standardized (mean 0, variance 1) normal random variable and  $\sigma^2(f)$  is the variance of  $Y_1(f) - r(f)M_1$ .

For calculation of theoretical values, we restrict attention to the mean passage time; thus, the function  $f$  in the definition of  $r(f)$  is the identity function. Using the results of Section 2.1, we show how to compute the value of the mean passage time  $r$  and the corresponding variance constant  $\sigma^2$  appearing in the central limit theorem of Equation (4.3). These calculations rest on the definition of two particular functions (denoted  $f^*$  and  $g^*$ ) having domain  $\bar{F}$  and taking values in the set  $\{0,1\}$ .

We define the function  $f^*$  to be the indicator function,  $1_S$ , of the set  $S$ ; i.e., for  $(z,n,z',n') \in \bar{F}$ ,

$$f^*(z,n,z',n') = 1_S(z,n,z',n'). \quad (4.4)$$



Proposition 4.1 follows directly from Equations (3.1) and (3.2).

PROPOSITION 4.1. Let  $f^*$  be the function defined by Equation (4.4), and  $R$  the transition matrix of the discrete time Markov chain  $\{V_k; k \geq 0\}$ . Then

$$E(Y_1(f^*)) = E\left\{\sum_{k=0}^{\delta_1-1} f^*(V_k)\right\} = (I - R)^{-1} f^*$$

and

$$E((Y_1(f^*))^2) = (I - R)^{-1} h^*,$$

where  $\delta_1$  is the time of the first return to the state 0 in  $\{V_k; k \geq 0\}$  and  $h^* = 2f^* \cdot E(Y_1(f^*)) - f^* \cdot f^*$ .

We use this result and the definition of  $M_1$  to obtain the quantities  $E_0(M_1)$  and  $E_0(M_1^2)$  according to

$$E_0(M_1) = E_0(Y_1(f^*)) \quad (4.5)$$

and

$$E_0(M_1^2) = E_0((Y_1(f^*))^2). \quad (4.6)$$

For an element  $(z, n, z', n') \in F$ , the value of the function  $g^*$  is 1 if a passage time for the marked job starts or is underway when the configuration of the job stack is  $z'$  and the marked job is in position  $n'$ , and is 0 otherwise. Formally, let  $D$  be the state space of the process  $Z = \{Z(t); t \geq 0\}$ , and denote by  $C$  the set of (center, class) pairs in the network. We define a function  $h$  taking values in  $C$  and having domain  $\bigcup_{z \in D} \{1, 2, \dots, N\}$  as follows. For  $z \in D$  and  $1 \leq n \leq N$ , the value of  $h(z, n)$

is  $(i, j)$  when the job in position  $n$  in the job stack  $z$  is of class  $j$  at center  $i$ . Now consider the embedded jump chain  $\{V_k: k \geq 0\}$  associated with the continuous time Markov chain  $\underline{V}$ . For states  $v', v'' \in F$ , the state space of  $\{V_k: k \geq 0\}$ , we write  $v' \sim v''$  when  $v''$  is accessible from  $v'$ , i.e., when for some  $n \geq 1$ , the probability starting from  $v'$  of entering  $v''$  on the  $n$ th step is positive. Similarly, for any subset  $L$  of  $F$  we write  $v' \overset{L}{\sim} v''$  when  $v''$  is accessible from  $v'$  under the taboo  $L$ . This means (cf. Chung (1967), pp. 43, 48) that for some  $n \geq 1$ , there is a positive probability, starting from state  $v'$ , of entering  $v''$  on the  $n$ th step under the restriction that none of the states in  $L$  is entered in between (exclusive of both ends).

Denoting the set of (center, class) pairs in the network by  $C$ , we define a subset  $G$  of  $C$  according to

$$G = \{(i, j) \in C: \text{ for some } (z, n, z', n') \in S, h(z', n') = (i, j)\} \cup \\ \{(i, j) \in C: \text{ for some } (z, n, z', n') \in F - (S \cup T), v' \in S \text{ and } v'' \in T, \\ v' \overset{T}{\sim} (z, n, z', n'), (z, n, z', n') \overset{T}{\sim} v'' \text{ and } h(z', n') = (i, j)\}.$$

Thus, the set  $G = G_1 \cup G_2$ , where a (center, class) pair is in the set  $G_1$  [resp.  $G_2$ ] if it is possible for the marked job to be of class  $j$  at center  $i$  when the passage time specified by the sets  $A_1, A_2, B_1$ , and  $B_2$  starts [resp. is underway].

Now, for  $(z, n, z', n') \in F$ , we define the function  $g^*$  as

$$g^*(z, n, z', n') = i_G(h(z', n')) . \quad (4.7)$$

Then we have

PROPOSITION 4.2. Let  $g^*$  be the function defined by Equation (4.7), and  $R$  be the jump matrix and  $q$  the vector of rate parameters for holding times in the continuous time Markov chain  $\underline{Y}$ . Then

$$E\{Y_1(g^*)\} = E\left\{\int_0^{\beta_1} g^*(V(s))ds\right\} = (I - {}_0R)^{-1}(g^* \cdot q^{-1}),$$

and

$$E\{(Y_1(g^*))^2\} = (I - {}_0R)^{-1}(h^* \cdot q^{-1}),$$

where  $\beta_1$  is the time of the first return to the state 0 in  $\underline{Y}$ , and  $h^* = 2f^* \cdot E\{Y_1(g^*)\}$ .

Proposition 4.2 follows directly from Equations (3.3) and (3.4). We use this result together with the observation that

$$\int_0^{\beta_1} g^*(V(s))ds = \sum_{j=1}^{M_1} P_j, \quad (4.8)$$

to obtain the quantities

$$E_0\left\{\sum_{j=1}^{M_1} P_j\right\} = E_0\{Y_1(g^*)\} \quad (4.9)$$

and

$$E_0\left\{\left(\sum_{j=1}^{M_1} P_j\right)^2\right\} = E_0\{(Y_1(g^*))^2\}. \quad (4.10)$$

Using the ratio formula, Equations (4.5) and (4.9) yield the quantity  $r$ . To obtain an expression for the variance constant  $\sigma^2$  appearing in the central limit theorem (Equation (4.3)) for the marked job method, we require one additional result.

PROPOSITION 4.3. Let  $R$  be the jump matrix and  $q$  the vector of rate parameters for holding times in the continuous time Markov chain  $\underline{V}$ . For the functions  $f^*$  and  $g^*$  defined by Equations (4.4) and (4.7),

$$E \left\{ \int_0^{\delta_1} g^*(V(s)) ds \sum_{k=0}^{\delta_1-1} f^*(V_k) \right\} = (I - {}_0R)^{-1} h^*,$$

where  $h^* = [(I - {}_0R)^{-1} f^*] \circ g^* \circ q^{-1} + [(I - {}_0R)^{-1} (g^* \circ q^{-1})] \circ f^* - g^* \circ q^{-1} \circ f^*$ .

The proof of Proposition 4.3. is in the Appendix.

We use this result to obtain

$$E_0 \left\{ \left( \sum_{j=1}^{M_1} p_j \right) M_1 \right\} = E_0 \left\{ \int_0^{\delta_1} g^*(V(s)) ds \sum_{k=0}^{\delta_1-1} f^*(V_k) \right\}. \quad (4.11)$$

Then an expression for the variance constant

$$\sigma^2 = E_0 \left\{ \left( \sum_{j=1}^{M_1} p_j \right)^2 \right\} - 2r E_0 \left\{ \left( \sum_{j=1}^{M_1} p_j \right) M_1 \right\} + r^2 E_0 (M_1^2),$$

follows from Equations (4.6), (4.10), and (4.11).



When comparing the statistical efficiency of the marked job and decomposition methods, it is convenient to have a central limit theorem comparable to Equation (4.3) but in terms of simulation time,  $t$ , rather than number of cycles,  $n$ . Let  $m(t)$  be the number of passage times completed by time  $t$ , i.e., in the interval  $(0, t]$ . If we denote by  $n(t)$  the number of 0-cycles completed by time  $t$ , then from renewal theory, as  $t \rightarrow \infty$ ,

$$\frac{n(t)}{t} \rightarrow \frac{1}{E_0\{\alpha_1\}}$$

with probability one, where  $E_0\{\alpha_1\}$  is the expected length of a 0-cycle in  $\mathcal{Y}$ . This implies that for large  $t$ , the number of 0-cycles completed by time  $t$  is approximately  $t/E_0\{\alpha_1\}$ . Combining this result with Equation (4.3), it follows that as  $t \rightarrow \infty$ ,

$$\frac{t^{1/2} \left[ \{m(t)\}^{-1} \left( \sum_{i=1}^{m(t)} f(P_i) \right) - r(f) \right]}{(E_0\{\alpha_1\})^{1/2} \sigma(f)/E_0\{M_1\}} \rightarrow N(0,1) .$$

This result is independent of the initial state  $V(0)$ . Since the numerator in this central limit theorem is independent of the state 0 selected to form cycles, so is the denominator. Thus for the mean passage time,

$$e = (E_0\{\alpha_1\})^{1/2} \sigma/E_0\{M_1\} \quad (4.12)$$

is the appropriate measure of statistical efficiency for the marked job method and is independent of the state 0 selected to form cycles. Note that we obtain the quantity  $E_0\{\alpha_1\}$  according to

$$E_0\{\alpha_1\} = E_0\{Y_1(1)\} ,$$

where  $1$  is the  $f$  function identically equal to one and

$$E\{Y_1(1)\} = E\left\{\int_0^{\beta_1} 1(V(s))ds\right\} = (I - {}_0R)^{-1}(L \circ q^{-1}) .$$

##### 5. VARIANCE CONSTANTS FOR THE DECOMPOSITION METHOD

We now turn to the decomposition method. As in Shedler (1979) we label the jobs from 1 to  $N$ , and for  $i=1,2,\dots,N$ , denote by  $N^i(t)$  the position of job  $i$  in the linear job stack at time  $t$ . Then, in terms of the vector  $Z(t)$  corresponding to the job stack, we set

$$X^0(t) = (Z(t), N^1(t), N^2(t), \dots, N^N(t)) .$$

The process  $X^0 = \{X^0(t) : t \geq 0\}$  is an irreducible, positive recurrent continuous time Markov chain with state space  $E^0$ .

We let  $L^0(t)$  denote the last state visited by the Markov chain  $X^0$  before jumping to  $X^0(t)$ , and for  $t \geq 0$  define

$$V^0(t) = (L^0(t), X^0(t)) .$$

The process  $V^0 = \{V^0(t) : t \geq 0\}$  is the fundamental stochastic process of the passage time simulation.

We denote the state space of  $V^0$  by  $F^0$  and define the subsets  $S^0$  and  $T^0$  of  $F^0$  according to

$$S^0 = \{(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) \in F^0 : \text{for some } k, \\ (z, n_k) \in A_1, \text{ and } (z', n'_k) \in A_2\}$$

and



$$T^0 = \{(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) \in F^0: \text{ for some } k, \\ (z, n_k) \in B_1, \text{ and } (z', n'_k) \in B_2\}.$$

The successive entrances of  $\gamma^0$  to the fixed subset  $S^0$  of  $F^0$  correspond to the starts of passage times irrespective of job identity, and the entrances of  $\gamma^0$  to the subset  $T^0$  correspond to the terminations.

The decomposition method applies to passage times for which the sets  $S$  and  $T$  (which define the starts and terminations of passage times) are disjoint. Denoting by  $\{P_n^0: n \geq 1\}$  the sequence of passage times (irrespective of job identity), enumerated in order of passage time start, by the argument in the Appendix of [9],  $P_n^0 \Rightarrow P_n^0$ , and the goal of the simulation is estimation of

$$r^0(f) = E\{f(P^0)\}.$$

To estimate  $r^0(f)$ , we carry out the simulation of the process  $\gamma^0$  in random blocks defined by the successive entrances of the process to the fixed set of states  $U^0$ . Entrances of  $\gamma^0$  to the set  $U^0$  correspond to the terminations of passage times (irrespective of job identity) which occur when no other passage times are underway, and which leave a fixed configuration of the job stack. Formally, recall that  $C$  is the set of (center, class) pairs in the network and define a subset  $H$  of  $C$  according to

$$H = \{(i, j) \in C: \text{ for some } (z, n, z', n') \in T-S, h(z', n') = (i, j)\} \cup \\ \{(i, j) \in C: \text{ for some } (z, n, z', n') \in F-(S \cup T), v' \in T \text{ and } v'' \in S, \\ v' \xrightarrow{S} (z, n, z', n'), (z, n, z', n') \xrightarrow{S} v'' \text{ and } h(z', n') = (i, j)\}.$$

Thus, the set  $Z = H_1 \cup H_2$  where a (center, class) pair is in the set  $H_1$  [resp.  $H_2$ ] if it is possible for the marked job to be of class  $j$  at center  $i$  when a passage time for the marked job specified by the sets  $A_1$ ,  $A_2$ ,  $B_1$ , and  $B_2$  terminates [resp. is not underway].

Now define a subset  $D^0$  of  $D$ , the state space of  $Z = \{Z(t) : t \geq 0\}$ , according to

$$D^0 = \{z \in D : h(z, n) \in H \text{ for } 1 \leq n \leq N \text{ and for some } n, h(z, n) \in H_1\}.$$

Elements of the set  $D^0$  correspond to configurations of the job stack upon termination of a passage time with no other passage times underway. The set  $D^0$  is nonempty because  $S \cap T \neq \emptyset$ , and thus the set  $H_1$  is nonempty. Therefore, we can select an element  $z^0$  of  $D^0$ , and (in terms of this fixed  $z^0$ ) define the set  $U^0$  according to

$$U^0 = \{(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) \in T^0 : z' = z^0\},$$

where  $T^0$  is the subset of  $F^0$  corresponding to the terminations of passage times.

For  $k \geq 1$ ,  $\gamma_k^0$  denotes the length in discrete time units of the  $k$ th block (returns to the set  $U^0$ ) of the process  $(\gamma_n^0 : n \geq 0)$ ; the latter is the embedded jump chain associated with  $\gamma^0$ , and we set  $\delta_0^0 = 0$  and  $\delta_n^0 = \gamma_n^0 + \dots + \gamma_1^0$ ,  $n \geq 1$ . We assume that  $\gamma^0(0) \in U^0$  and denote by  $K_n^0$  the number of passage times in the  $n$ th block of the process  $\gamma^0$ . Also, we let  $\gamma_n^0(f)$  be the sum of the quantities  $f(\tau^0)$  over the passage times in the  $n$ th block of  $\gamma^0$ . The key results of [14] leading to point estimates and confidence intervals for the quantity  $\tau^0(f)$  are that the pairs of random variables

$$\{(Y_m^0(f), K_m^0) : m \geq 1\} \quad (5.1)$$

are independent and identically distributed, and that

$$r^0(f) = E_{U^0}\{Y_1^0(f)\} / E_{U^0}\{K_1^0\}, \quad (5.2)$$

provided that the quantity  $E\{|f(P^0)|\} < \infty$ . The symbol  $E_{U^0}\{\cdot\}$  is an abuse of our previous notation. It connotes conditional expectation associated with starting the Markov chain  $\underline{Y}^0$  in one of the states in the set  $U^0$ . The definition of the set  $U^0$  implies that the conditional expectation is independent of the particular starting state in  $U^0$ .

Given Equations (5.1) and (5.2), from a fixed number of blocks of  $\underline{Y}^0$ , the decomposition method provides the point estimate

$$\hat{r}_n^0(f) = \bar{Y}_n^0(f) / \bar{K}_n^0,$$

and the associated confidence interval for  $r^0(f)$  follows from the central limit theorem

$$\frac{n^{1/2}(\hat{r}_n^0(f) - r^0(f))}{\sigma^0(f) / E_{U^0}\{K_1^0\}} \rightarrow N(0, 1). \quad (5.3)$$

The quantity  $(\sigma^0(f))^2$  is the variance of  $Y_1^0(f) - r^0(f)K_1^0$ .

Taking  $f$  to be the identity function, we restrict attention to the mean value  $r^0$  and consider the associated theoretical values. By the argument which leads to Equation (4.12), an appropriate measure of the statistical efficiency of the simulation is the quantity



$$e^0 = \left( \mathbb{E}_{U^0}(\alpha_1^0) \right)^{1/2} \sigma^0 / \mathbb{E}_{U^0}(\kappa_1^0), \quad (5.4)$$

where  $\alpha_1^0$  is the length of a block in the continuous time process  $\underline{y}^0$ .

The individual quantities required to compute this measure of efficiency are defined in terms of the successive returns of the process  $\underline{y}^0$  to a fixed set of states ( $U^0$ ) rather than to a single state. Moreover, the successive entrances of  $\underline{y}^0$  to  $U^0$  are not regeneration points for  $\underline{y}^0$ . Accordingly, it is not possible to apply the results of Section 2 directly, as we did for the marked job method. We can, however, select a fixed state (designated state  $u^0$ ) from the set  $U^0$  and compute the quantity corresponding to Equation (5.4) for the resulting  $u^0$ -cycles. (Note that the successive entrances of the process  $\underline{y}^0$  to the fixed state  $u^0$  are regeneration points for the process  $\underline{y}^0$ .) The expression in Equation (5.4) computed for  $u^0$ -cycles is

$$e^0(u^0) = \left( \mathbb{E}_{U^0}(\alpha_1^0) \right)^{1/2} \sigma_u^0 / \mathbb{E}_{U^0}(\kappa_1^0),$$

where the constant  $\sigma_u^0$  (analogous to  $\sigma^0$ ) is defined for  $u^0$ -cycles. This quantity  $e^0(u^0)$  is equal to  $e^0$ . To see this, for  $t \geq 0$  let  $n^0(t)$  be the number of passage times (irrespective of job identity) completed in the interval  $(0, t]$ . In terms of simulation time,  $t$ , we have the central limit theorem

$$t^{1/2} \left[ \left( \alpha^0(t) \right)^{-1} \left( \sum_{i=1}^{n^0(t)} f(\beta_i^0) \right) - \tau^0(f) \right] / \left[ \left( \mathbb{E}_{U^0}(\alpha_1^0) \right)^{1/2} \sigma_u^0(f) / \mathbb{E}_{U^0}(\kappa_1^0) \right] \rightarrow N(0,1),$$

and, when  $f$  is the identity function, the variance constant in the denominator is the quantity  $e^0$ . There is a similar central limit theorem

in terms of  $u^0$ -cycles; the numerator is the same and the variance constant in the denominator is  $e^0(u^0)$ . Since the numerators in these two central limit theorems are the same, as are the limiting random variables  $(N(0,1))$ ,  $e^0$  must equal  $e^0(u^0)$ . For a similar argument, see Propositions 5.1 and 5.6 of Crane and Iglehart (1975).

Next we observe that the number of passage times in a  $u^0$ -cycle of the process  $\underline{Y}^0$ , as well as the sum of the passage times in a  $u^0$ -cycle, does not depend on the identities of the jobs in successive configurations of the job stack. It follows that we can work with the process  $\underline{W}=\{W(t):t\geq 0\}$  defined by

$$W(t) = (K(t), Z(t)) .$$

Here  $Z(t)$  corresponds to the linear job stack, and  $K(t)$  is the last state visited by the Markov chain  $\underline{Z}=\{Z(t):t\geq 0\}$  before jumping to  $Z(t)$ . The process  $\underline{W}$  is an irreducible, positive recurrent continuous time Markov chain with a state space that is a subset of  $D \times D$ . Note that in general the state space of  $\underline{W}$  is much smaller than that of  $\underline{Y}^0$ , and that working with the process  $\underline{W}$  is computationally advantageous.

The computations rest on the definition of two particular functions ( $f$  and  $g$ ) defined on the state space of  $\underline{W}$  and taking values in the set  $\{0,1\}$ . We define the functions  $f$  and  $g$  in terms of functions  $f^0$  and  $g^0$  defined on  $F^0$ , the state space of the process  $\underline{Y}^0$ . We take the function  $f^0$  to be the indicator function,  $1_{S^0}$ , of the set  $S^0$  which defines the starts

of passage times irrespective of job identity; i.e., for

$$(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) \in F^0,$$

$$f^0(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) = 1_{S^0}(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) .$$

Thus if a passage time for some job starts when  $y^0$  hits

$(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N)$ , then  $f^0 = 1$ . Note that for each  $(z, z')$  in the state space of  $\bar{N}$ , there exist  $n_1, \dots, n_N, n'_1, \dots, n'_N$  such that  $(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) \in F^0$ . For such a  $(z, z')$ , we define

$$f(z, z') = f^0(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) . \quad (5.5)$$

The function  $f$  is well defined since, for fixed  $z$  and  $z'$ , the function  $f^0$  is independent of its other arguments.

For an element  $(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) \in F^0$ , the value of the function  $g^0$  is the number of passage times that start or are underway when the configuration of the job stack is  $z'$ . Formally, for  $(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) \in F^0$ , we define

$$g^0(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) = \sum_{k=1}^N 1_G(h(z', k)) .$$

Then, for  $(z, z')$  in the state space of  $\bar{N}$ , we set

$$g(z, z') = g^0(z, n_1, \dots, n_N, z', n'_1, \dots, n'_N) . \quad (5.6)$$

The justification for using the process  $\bar{N}$  is that the number of passage times (which start and terminate) in the first  $u^0$ -cycle of  $y^0$  is



$$\sum_{j=0}^{\kappa_1-1} f(W_j) ,$$

and the sum of the passage times in the first  $u^0$ -cycle of  $y^0$  is

$$\int_0^{\zeta_1} g(W(s)) ds .$$

Here  $\zeta_1$  [resp.  $\kappa_1$ ] is the time of the first return of the process  $\underline{W}$  [resp. the jump chain  $\{W_k: k \geq 0\}$ ] to the fixed state  $w^0$ . The return state  $w^0$  corresponds to the fixed state  $u^0$  selected from the set  $U^0$ , i.e., if

$$u^0 = (z, n_1, \dots, n_N, z^0, n'_1, \dots, n'_N) ,$$

then  $w^0 = (z, z^0)$ .

By direct application of Equations (3.1) and (3.2), we have

PROPOSITION 5.1. Let  $f$  be the function defined by Equation (5.5), and  $R$  be the transition matrix of the discrete time Markov chain  $\{W_k: k \geq 0\}$ . Then

$$E\{Y_1(f)\} = E\left\{\sum_{k=0}^{\kappa_1-1} f(W_k)\right\} = (I - {}_0R)^{-1}f$$

and

$$E\{(Y_1(f))^2\} = (I - {}_0R)^{-1}h ,$$

where  $\kappa_1$  is the time of the first return to the state  $w^0$  in  $\{W_k: k \geq 0\}$  and  $h = 2f \circ E(Y_1(f)) - f \circ f$ .

From Proposition (3.1) we obtain  $E_{u^0}(K_1^0)$  and  $E_{u^0}((K_1^0)^2)$  according to

$$E_{u^0}(K_1^0) = E_{w^0}(Y_1(z)) \quad (5.7)$$

and

$$E_{u^0}((K_1^0)^2) = E_{w^0}((Y_1(z))^2) . \quad (5.8)$$

By direct application of Equations (3.3) and (3.4), we have

PROPOSITION 5.2. Let  $g$  be the function defined by Equation (3.6), and  $R$  be the jump matrix and  $q$  the vector of rate parameters for holding times in the continuous time Markov chain  $\tilde{Y}$ . Then

$$E(Y_1(z)) = E\left\{\int_0^{\tau_1} g(W(s))ds\right\} = (I - R)^{-1}(g \cdot q^{-1}) ,$$

and

$$E((Y_1(z))^2) = (I - R)^{-1}(h \cdot q^{-1}) ,$$

where  $\tau_1$  is the time of the first return to the state  $w^0$  in  $\tilde{Y}$ , and  $h = E(g^2)$ .

We use this result to obtain

$$E_{u^0}\left\{\sum_{j=1}^{K_1^0} p_j^0\right\} = E_{w^0}(Y_1(z)) \quad (5.9)$$

and

$$E_{u^0}\left\{\left(\sum_{j=1}^{K_1^0} p_j^0\right)^2\right\} = E_{w^0}((Y_1(z))^2) . \quad (5.10)$$

Using the ratio formula, Equations (5.7) and (5.9) yield the value of  $r^0$ . Analogous to Proposition 4.3 we have

PROPOSITION 5.3. For  $E\{Y_1(f)\}$  and  $E\{Y_1(g)\}$  given by Propositions 5.1 and 5.2,

$$E\left\{\int_0^{\zeta_1} g(W(s))ds \sum_{j=0}^{\kappa_1-1} f(W_k)\right\} = (I - {}_0R)^{-1}h,$$

where  $h = (g \circ q^{-1}) \circ E\{Y_1(f)\} + f \circ E\{Y_1(g)\} - (g \circ q^{-1}) \circ f$ .

We use this result to obtain

$$E_{U^0}\left\{\left(\sum_{j=1}^{K_1^0} P_j^0\right) K_1^0\right\} = E_{W^0}\left\{\int_0^{\zeta_1} g(W(s))ds \sum_{k=0}^{\kappa_1-1} f(W_k)\right\}, \quad (5.11)$$

and an expression for the variance constant follows from Equations (5.8), (5.10), and (5.11).

## 6. NUMERICAL RESULTS

We consider the closed network of queues of Figure 1 and the limiting passage times  $P$  and  $R$  therein. Upon completion of service to a job at center 1, in accordance with a binary-valued variable  $\psi$ , the job joins the tail of the queue in center 1 (when  $\psi=1$ ) or joins the tail of the queue in center 2 (when  $\psi=0$ ). Neither center 1 nor center 2 service is subject to interruption. We assume that both centers provide service according to a FCFS (first-come, first-served) queueing discipline. The limiting passage time  $P$  is associated with the time measured from entrance by a job into the center 1 queue upon completion of a center 2 service until the job next enters the center 2 queue. Similarly, the response time  $R$  is associated

with the time between successive entrances of a job into the center 1 queue upon completion of a center 2 service.

We make the assumptions that (i) service times at centers 1 and 2 are mutually independent and (ii) successive service times at center  $i$  form a sequence of i.i.d. random variables exponentially distributed with rate parameter  $\lambda_i$ ,  $i=1,2$ . In addition, we assume that the routing variable  $\psi$  is a Bernoulli random variable, and values of  $\psi$  at successive completions of service at center 1 form a sequence of i.i.d. random variables, independent of the service times at centers 1 and 2. We denote by  $p$  the probability that the routing random variable  $\psi$  takes the value 1.

Since each center sees only one job class, by taking into account the fixed number of jobs in the network, for  $t \geq 0$  we can define  $Z(t)$  to be the number of jobs waiting or in service at center 1 at time  $t$ . Then the process  $X = \{(Z(t), N(t)) : t \geq 0\}$ , where  $N(t)$  is the position of the marked job in the job stack at time  $t$ , has state space

$$E = \{(i, j) : 0 \leq i \leq N; 1 \leq j \leq N\}.$$

For the passage time  $P$ , the sets  $A_1$  and  $A_2$  defining the starts of passage times are

$$A_1 = \{(1, N) : 0 \leq i \leq N\},$$

and

$$A_2 = \{(1, 1) : 0 \leq i \leq N\}.$$



Similarly, the sets  $B_1$  and  $B_2$  defining the terminations of the passage time  $P$  are

$$B_1 = \{(i, 1) : 0 < i \leq N\} ,$$

and

$$B_2 = \{(i-1, 1) : 0 < i \leq N\} .$$

For the response time  $R$ , the sets  $A_1$  and  $A_2$  are the same as for the passage time  $P$ , but  $B_1 = A_1$  and  $B_2 = A_2$ .

In connection with the marked job method, the continuous time process  $\underline{Y} = \{(L(t), X(t)) : t \geq 0\}$ , where  $L(t)$  is the last state visited by the Markov chain  $\underline{X}$  before jumping to  $\underline{X}(t)$ , has state space  $F$ , where

$$F = \{(i, j, i+1, j+1) : 0 \leq i < N, 1 \leq j < N\} \cup \{(i, N, i+1, 1) : 0 \leq i < N\} \cup$$

$$\{(i, j, i-1, j) : 0 < i \leq N, 1 \leq j \leq N\} \cup \{(i, i, i, 1) : 1 \leq i \leq N\} .$$

The subsets of  $F$  defining the starts and terminations of passage times for the marked job are

$$S = \{(i, N, i+1, 1) : 0 \leq i < N\}$$

and

$$T = \{(i, i, i-1, 1) : 0 < i \leq N\} .$$

Tables 1 and 2 give theoretical values for simulation of the closed network of queues by the marked job method. Numerical results are

displayed for the mean of the response time  $R$  and corresponding results for the passage time  $P$  are in parentheses. For the case of  $N=2$  jobs (Table 1), the set  $S=\{(0,2,1,1), (1,2,1,1)\}$ . With  $\lambda_1=1$ ,  $\lambda_2=0.5$ , and  $p=0.75$ , the numerical results show that on the average 0-cycles defined by returns to the state  $(0,2,1,1)$  are twice as long as those defined by the returns to the state  $(1,2,1,1)$ . Note that as expected, the quantities  $\sigma^2/E_0\{M_1\}$  (as well as  $(E_0\{x_1\})^{1/2}\sigma/E_0\{M_1\}$ ) are the same for the two return states. Table 2 gives results for  $N=4$  jobs. Here there are four possible return states, and for the parameter values selected, returns to the state  $(3,4,4,1)$  occur most frequently, and on the average eight times more often than returns to the state  $(0,4,1,1)$ .

We now turn to the decomposition method. The process

$$\tilde{x}^0 = \{(Z(t), N^1(t), \dots, N^V(t)) : t \geq 0\}$$

has state space  $E^0$ , where

$$E^0 = \{(i, n_1, \dots, n_N) : 0 \leq i \leq N; 1 \leq n_1, \dots, n_N \leq N; n_i \neq n_j \text{ for } i \neq j\}.$$

The underlying continuous time process  $\tilde{y}^0 = (V^0(t) : t \geq 0)$  defined by

$$\tilde{y}^0(t) = (L^0(t), X^0(t)),$$

where  $L^0(t)$  is the last state visited by the Markov chain  $\tilde{x}^0$  before jumping to  $X^0(t)$ , has state space  $F^0$ . The subsets of  $F^0$  defining starts and terminations of passage times are

$$S^0 = \{(i, n_1, \dots, n_N, i+1, n'_1, \dots, n'_N) : 0 \leq i < N; \text{ for exactly one } j^*,$$

$$n_{j^*} = N \text{ and } n'_{j^*} = 1; 1 \leq n_j < N \text{ and } n'_j = n_j + 1, j \neq j^*; n_k \neq n_j \text{ for } k \neq j\}$$

and

$$T^0 = \{(i, n_1, \dots, n_N, i-1, n'_1, \dots, n'_N) : 0 < i \leq N;$$

$$1 \leq n_j = n'_j \leq N, \text{ for } 1 \leq j \leq N; n_k \neq n_j \text{ for } k \neq j\}.$$

The process  $Z = \{Z(t) : t \geq 0\}$  has state space  $D = \{0, 1, \dots, N\}$ , the set  $D^0 = \{0\}$ , and the set  $U^0$  defining cycles of the process  $Y^0$  is

$$U^0 = \{(1, n_1, \dots, n_N, 0, n_1, \dots, n_N) : 1 \leq n_1, \dots, n_N \leq N; n_i \neq n_j \text{ for } i \neq j\}.$$

The state space of the stochastic process  $\tilde{W}$  is

$$\{(i, i+1) : 0 \leq i \leq N-1\} \cup \{(i, i-1) : 1 \leq i \leq N\},$$

and the state  $w^0 = (1, 0)$ .

Table 3 gives theoretical values for simulation of the closed network of queues by the decomposition method for the quantity  $E(P)$ . The table gives results for  $N=1$  to  $N=4$  jobs, and the parameter values are the same as in Tables 1 and 2. For  $N=2$  jobs, the value of the quantity  $e^0(u^0) = e^0$  which measures the statistical efficiency of the decomposition method is 16.546. The corresponding value from Table 1 for the marked job method is 20.390. Thus, for these parameter values the decomposition method is approximately 21 percent more efficient than the marked job method. For  $N=4$  jobs, the decomposition method is 41 percent more efficient.



Numerical results pertaining to the statistical efficiency of the decomposition method in simulation of the closed network of queues appear in Table 4. For  $N=1$  to  $N=6$  jobs, the table gives theoretical values of the quantities  $r^0$  and  $e^0$  for three sets of parameters values. We hold the value of  $\lambda_1=1$  and  $p=0.75$  fixed, but vary  $\lambda_2$ . Table 5 gives a comparison of the statistical efficiency of the marked job and decomposition methods for the same sets of parameter values.



TABLE 1

Theoretical Values for the Marked Job Method.  
 Passage Time R (P) in Closed Network of Queues.  
 $N=2$ ,  $\lambda_1=1.0$ ,  $\lambda_2=0.5$ ,  $p=0.75$ .

Parameter	Return State of $\underline{V}=\{V(t):t \geq 0\}$	
	(0,2,1,1)	(1,2,1,1)
$E_0\{\alpha_1\}$	24.0	12.0
$E_0\left\{\sum_{j=1}^{M_1} P_j\right\}$	28.0 (20.0)	14.0 (10.0)
$E_0\{M_1\}$	3.0 (3.0)	1.5 (1.5)
$E_0\left\{\sum_{j=1}^{M_1} P_j\right\} / E_0\{M_1\}$	9.333 (6.667)	9.333 (6.667)
$\sigma^2$	140.267 (129.067)	70.133 (64.533)
$\sigma^2 / E_0\{M_1\}$	46.756 (43.022)	46.756 (43.022)
$(E_0\{\alpha_1\})^{1/2} \sigma / E_0\{M_1\}$	20.890 (20.038)	20.890 (20.038)

TABLE 2

Theoretical Values for the Marked Job Method.  
 Passage Time  $P$  (R) in Closed Network of Queues.  
 $N=4$ ,  $\lambda_1=1.0$ ,  $\lambda_2=0.5$ ,  $p=0.75$ .

Parameter	Return State of $\underline{V}=\{V(t):t \geq 0\}$			
	(0,4,1,1)	(1,4,2,1)	(2,4,3,1)	(3,4,4,1)
$E_0(\alpha_1)$	216.0	108.0	54.0	27.0
$E_0 \left\{ \sum_{j=1}^{M_1} P_j \right\}$	248.0 (196.0)	124.0 (98.0)	62.0 (49.0)	61.0 (24.5)
$E_0(M_1)$	15.0 (13.0)	7.5 (7.5)	3.75 (3.75)	1.875 (1.875)
$E_0 \left\{ \sum_{j=1}^{M_1} P_j \right\} / E_0(M_1)$	16.533 (13.067)	16.533 (13.067)	16.533 (13.067)	16.533 (13.067)
$\sigma_0^2$	2111.343 (2139.600)	1055.672 (1069.800)	527.836 (534.900)	263.918 (267.450)
$\sigma_0^2 / E_0(M_1)$	140.756 (142.640)	140.756 (142.640)	140.756 (142.640)	140.756 (142.640)
$(E_0(\alpha_1))^{1/2} \sigma_0 / E_0(M_1)$	48.241 (48.562)	48.241 (48.562)	48.241 (48.562)	48.241 (48.562)

TABLE 3

Theoretical Values for the Decomposition Method.  
 Passage Time P in Closed Network of Queues.  
 $\lambda_1=1.0$ ,  $\lambda_2=0.5$ ,  $p=0.75$ .

Parameter	N=1	N=2	N=3	N=4
$E_{u0}\{\alpha_1^0\}$	6.0	24.0	30.0	62.0
$E_{u0}\left\{\sum_{j=1}^{K_1^0} P_j^0\right\}$	4.0	20.0	68.0	196.0
$E_{u0}\{K_1^0\}$	1.0	3.0	7.0	15.0
$E_{u0}\left\{\sum_{j=1}^{K_1^0} P_j^0\right\}/E_{u0}\{K_1^0\}$	4.0	6.667	9.714	13.067
$(\sigma_0^0)^2$	16.0	176.0	1023.673	4317.227
$(\sigma_0^0)^2/E_{u0}\{K_1^0\}$	16.0	58.667	146.249	287.815
$(E_{u0}\{\alpha_1^0\})^{1/2} \sigma_0^0/E_{u0}\{K_1^0\}$	9.798	16.546	25.035	34.491



TABLE 4

Statistical Efficiency of the Decomposition Method.  
 Passage Time  $P$  in Closed Network of Queues.

	$p = 0.75$ $\lambda_1 = 1.0$ $\lambda_2 = 0.125$		$p = 0.75$ $\lambda_1 = 1.0$ $\lambda_2 = 0.25$		$p = 0.75$ $\lambda_1 = 1.0$ $\lambda_2 = 0.5$	
N	$r^0$	$a^0$	$r^0$	$a^0$	$r^0$	$a^0$
1	4.0	13.856	4.0	11.314	4.0	9.798
2	5.333	19.956	6.0	17.664	6.667	16.546
3	6.286	27.380	8.0	26.128	9.714	25.035
4	6.933	35.139	10.0	36.606	13.067	34.491
5	7.355	42.597	12.0	49.107	16.645	44.296
6	7.619	49.068	14.0	63.645	20.381	54.021



TABLE 5

Relative Efficiency of the Marked Job and Decomposition Methods.  
 Passage Time P in Closed Network of Queues.

N	$p = 0.75$ $\lambda_1 = 1.0$ $\lambda_2 = 0.125$	$p = 0.75$ $\lambda_1 = 1.0$ $\lambda_2 = 0.25$	$p = 0.75$ $\lambda_1 = 1.0$ $\lambda_2 = 0.5$
1	1.0	1.0	1.0
2	1.190	1.189	1.211
3	1.207	1.224	1.319
4	1.190	1.209	1.408
5	1.176	1.186	1.499
6	1.394	1.162	1.597

## APPENDIX

This appendix is devoted to a proof of the result given in Equation (3.5). Recall that  $\underline{X} = \{X(t); t \geq 0\}$  is a continuous state Markov chain with finite state space  $E = \{0, 1, 2, \dots, N\}$ . We assume  $\underline{X}$  is irreducible and hence positive recurrent. The mean holding time in state  $i$  is  $q_i^{-1}$  and the embedded jump chain  $\{X_n; n \geq 0\}$  has transition matrix  $R$ . Again  $f$  and  $g$  are real-valued functions defined on  $E$  and we view them as column vectors; e.g.,  $f = (f(0), \dots, f(N))$ . We take 0 as the return state and as before let  $\beta_1$  denote the time of first entrance of  $\underline{X}$  to 0 and  $\delta_1$  denote the time of first entrance of  $\{X_n; n \geq 0\}$  to 0. Let  $X(0) = 0$ ,  $\tau_0 = 0$ , and  $\tau_n$  ( $n \geq 1$ ) be the random time at which the  $n$ th transition of  $\underline{X}$  takes place. Since  $\underline{X}$  is irreducible and  $E$  is finite,  $\tau_n \rightarrow \infty$  with probability one.

PROPOSITION. Under the conditions above

$$E \left\{ \int_0^{\beta_1} f(X(s)) ds \sum_{n=0}^{\delta_1-1} g(X_n) \right\} = (I - R)^{-1} f, \quad (A.1)$$

where

$$\pi = \frac{1}{N} [(I - R)^{-1} g] \cdot f \cdot q^{-1} + [(I - R)^{-1} (f \cdot q^{-1})] \cdot g - f \cdot q^{-1} \cdot g.$$

Proof. First we decompose the expression in Equation (A.1) as follows:

$$\begin{aligned}
 \int_0^{\beta_1} f(X(s)) ds \sum_{n=0}^{\delta_1-1} g(X_n) &= \sum_{n=0}^{\delta_1-1} f(X_n) (\tau_{n+1} - \tau_n) \sum_{m=0}^{\delta_1-1} g(X_m) \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(X_n) (\tau_{n+1} - \tau_n) 1_{\{\delta_1 > n\}} g(X_m) 1_{\{\delta_1 > m\}} \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^n a_n b_m + \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} a_n b_m - \sum_{n=0}^{\infty} a_n b_n, \quad (A.2)
 \end{aligned}$$

where  $a_n$  and  $b_m$  are defined in the obvious way. Now take expectations on both sides of Equation (A.2). We compute the three expectations on the right-hand side of Equation (A.2) separately. The interchanges below of  $E_1$  and  $\sum$  can be justified since the process  $X$  is positive recurrent and the state space is finite. First the easy term:

$$\begin{aligned}
 E_1 \left( \sum_{n=0}^{\infty} a_n b_n \right) &= \sum_{n=0}^{\infty} E_1 \left\{ f(X_n) g(X_n) 1_{\{\delta_1 > n\}} (\tau_{n+1} - \tau_n) \right\} \\
 &= \sum_{n=0}^{\infty} E_1 \left\{ E_1 \left[ f(X_n) g(X_n) 1_{\{\delta_1 > n\}} (\tau_{n+1} - \tau_n) \mid X_j : j \leq n \right] \right\} \\
 &= \sum_{n=0}^{\infty} E_1 \left\{ f(X_n) g(X_n) 1_{\{\delta_1 > n\}} E_1 (\tau_{n+1} - \tau_n \mid X_j : j \leq n) \right\} \\
 &= \sum_{n=0}^{\infty} E_1 \left\{ f(X_n) g(X_n) 1_{\{\delta_1 > n\}} q_{X_n}^{-1} \right\} \\
 &= E_1 \left\{ \sum_{n=0}^{\delta_1-1} f(X_n) q_{X_n}^{-1} g(X_n) \right\}.
 \end{aligned}$$

From Equation (3.1) we can conclude that

$$E \left\{ \sum_{n=0}^{\infty} a_n b_n \right\} = (I - {}_0R)^{-1} (f \circ q^{-1} \circ g) . \quad (A.3)$$

The first term on the right-hand side of Equation (A.2) is handled in a similar fashion. After conditioning on  $\{X_j : j \leq n\}$  as above, we have

$$\begin{aligned} E_1 \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^n a_n b_m \right\} &= \sum_{n=0}^{\infty} \sum_{m=0}^n E_1 \left\{ f(X_n) q^{-1}(X_n) 1_{\{\delta_1 > n\}} g(X_m) 1_{\{\delta_1 > m\}} \right\} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n E_1 \{ c_n d_m \} . \end{aligned}$$

Now interchange the sums and make a change of variables to obtain

$$\begin{aligned} E_1 \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^n a_n b_m \right\} &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} E_1 \{ c_n d_m \} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} E_1 \{ c_{n+m} d_m \} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} {}_0r_{1j}^m {}_0r_{jk}^n g(j) f(k) q^{-1}(k) . \end{aligned}$$

Since  $\sum_{n=0}^{\infty} {}_0R^n = (I - {}_0R)^{-1}$ , we conclude that

$$E \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^n a_n b_m \right\} = (I - {}_0R)^{-1} \left[ (I - {}_0R)^{-1} (f \circ q^{-1}) \right] \circ g . \quad (A.4)$$

The second term on the right-hand side of Equation (A.2) requires a somewhat different conditioning argument.



$$\begin{aligned}
E_1 \left\{ \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} a_n b_m \right\} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} E_1 \{ a_n b_{m+n} \} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} E_1 \left\{ E_1 \{ a_n b_{m+n} | X_j : j \leq m+n \} \right\} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} E_1 \left\{ f(X_n) 1_{\{\delta_1 > n\}} q^{-1}(X_n) g(X_{m+n}) 1_{\{\delta_1 > m+n\}} \right\} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j \in E} \sum_{k \in E} o_{ij}^n o_{jk}^m f(j) q^{-1}(j) g(k) .
\end{aligned}$$

Proceeding as in Equation (A.4) yields

$$E \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} a_n b_m = (I - O^R)^{-1} \left( \left[ (I - O^R)^{-1} g \right] \circ f \circ q^{-1} \right) . \quad (A.5)$$

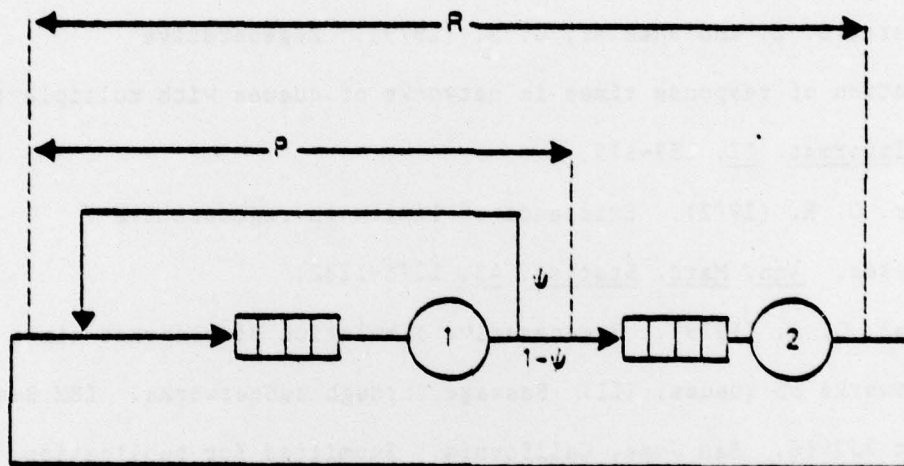
Combining Equations (A.3)-(A.5), we have Equation (A.1).

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- (i) Service times at centers 1 and 2 are not interruptable
- (ii) Routing determined by binary valued random variable  $\psi$

Figure 1. Closed network of queues.



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## 20. ABSTRACT

In recent papers we have provided estimation methods for general characteristics of "passage times" in certain closed networks of queues. These estimation procedures are based on the regenerative method for simulation analysis. Informally, a passage time is the time for a job to traverse a portion of the network. Such quantities are important in computer and communication system models, and in this context, quantities other than mean values are of interest. From a single simulation run, our passage time simulation methods provide both point estimates and confidence intervals.

We consider here the calculation of variance constants entering into central limit theorems used to obtain confidence intervals from passage time simulations. Using results of Hordijk, Iglehart, and Schassberger for the calculation of moments in discrete time and continuous time Markov chains, we calculate variance constants pertinent to mean passage times. We do this first for the "marked job method" for passage time simulation which is based on the tracking of a distinguished job, and then for the "decomposition method" in which observed passage times for all of the jobs enter into the construction of point and interval estimates. The results of this paper provide a means of comparing the statistical efficiency of the two estimation methods.

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